MATH 2230 B 03/02/2020

1. Analyticity

Def: If f is analytic at z_0, then f is differentiable in some neighborhood of z_0.

And if f is analytic in S, then f is analytic at each point of S.

exp. $f = xy + iy$.

1. To show Ref & Imf are C': f is C'.

2. $$\begin{cases} u = xy \\ v = y \end{cases} \Rightarrow \begin{cases} \begin{cases} u_x = y \\ v_y = 1 \end{cases} \\ \begin{cases} u_y = x \\ v_x = 0 \end{cases} \end{cases} \Rightarrow \begin{cases} y = 1 \\ x = 0 \end{cases}.$$

f is differentiable at i.

But in any neighborhood containing i, f fails to be differentiable, f is nowhere analytic.

Analyticity is stronger than differentiability.

Thm: If $f' = 0$ over domain D, f is analytic, then $f \equiv C$.

Proof: Show $\{f = f(z_0)\}$ is both open & closed. $\square$. (Basic idea).

Eg. If f is real-valued & analytic on D, then f is constant.

Proof: $u = \text{Re} f$, $v = \text{Im} f \equiv 0$.

So $\begin{cases} u_x = v_y = 0 \\ u_y = -v_x = 0 \end{cases} \Rightarrow u \equiv C$

$\Rightarrow f \equiv C \in \mathbb{R}$. $\square$.

2. Harmonic Function

C-R for analytic $f = u + iv$.

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \Rightarrow \begin{cases} u_{xx} = v_{yx} = v_{xy} \\ u_{yy} = -v_{xy} \end{cases}$$

$\Rightarrow u_{xx} + u_{yy} = 0$ similar for $v = \operatorname{Im} f$.

$\Delta = \partial_{xx} + \partial_{yy} \rightarrow$ Laplace Equation.

Def: If $\Delta u = 0$, then we say u is harmonic.

Given harmonic u on the plane, can find harmonic v s.t. $f = u + iv$ is analytic? (We call v harmonic conjugate.)

In general, no.

But on a simply connected domain,

the existence of harmonic conjugate is $\Leftrightarrow$ domain is simply connected.

3. Integral of $\mathbb{C}$-valued $w(t)$ on $[0,t]$.

$w(t) = u(t) + iv(t)$.

derivative

$w'(t) = u'(t) + iv'(t)$

Integral

$\int w(t) = \int u(t) + i \int v(t)$.

$$\begin{cases} Re \int w(t) = \int Re\, w(t) \\ Im \int w(t) = \int Im\, w(t). \end{cases}$$

Exp:
$$\int_0^{\frac{\pi}{4}} e^{it} = \frac{1}{i} e^{it} \Big|_0^{\frac{\pi}{4}} = e^{\frac{\pi}{4} i} - 1$$

$$= \int_0^{\frac{\pi}{4}} \cos t + i \sin t$$

$$= \sin t \Big|_0^{\frac{\pi}{4}} + i(-\cos t)\Big|_0^{\frac{\pi}{4}}$$

$w(t)$ to be C^1.

4. Contour.

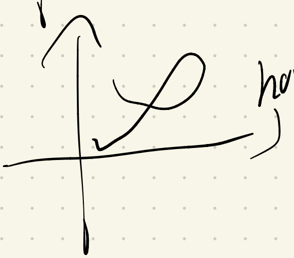
Def: Suppose an arc does not cross itself, then arc is called simple or Jordan arc.

When we say simple closed curve or Jordan curve, the are only intersect itself at the end pts of its parametrization.

An arc is $\{(x(t), y(t)) \mid t \in [a, b]\}$ and x, y are continuous.

Simple arc

Jordan Curve



not simple.

For an arc γ, suppose we have a parametrization

$$L(\gamma) = \int_a^b |\gamma'(t)|\, dt \qquad \gamma_0 : [a,b] \to \mathbb{C}$$

If we have another parametrization by $t : [c,d] \to [a,b]$, strictly increasing

$\gamma_1(s) = \gamma_0(t(s))$.

$$L = \int_c^d |\gamma_0'(t(s))\, t'(s)|\, ds$$

$$= \int_c^d |\gamma_0'(t(s))|\, t'(s)\, ds$$

$z = t(s)$.

$$= \int_a^b |\gamma_0'(z)|\, dz.$$

Contour: an arc consisting of finite number of smooth arcs joined end to end.

$z(t)$ be the parametrization of the contour.

Simple closed contour: only the initial & final values of $z(t)$ are the same.

J. Contour integral.

$$\int_\gamma f\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt.$$

Let $\gamma : [a,b] \to \mathbb{C}$, f is continuous.

$$\left|\int_\gamma f\,dz\right| \le \int_\gamma |f|\,|dz| = \max|f| \int_\gamma |dz|$$

$$= \max|f|\, L(\gamma).$$

So this integral is not infinite.

Another parametrization does not affect the integral.

Conformal property of analytic map.

f is analytic, 2 arcs $\gamma_1(t)$, $\gamma_2(t)$

$\boxed{\gamma_1(t) = \gamma_2(t)}$ at t_0 $\quad f(\gamma_1(t_0)) = f(\gamma_2(t_0))$.

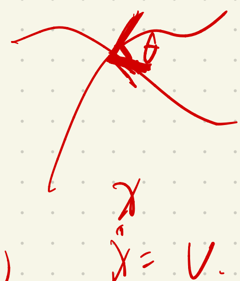
$$\frac{d}{dt} f(\gamma_i(t)) = \boxed{f'(\gamma_i(t))\,\gamma_i'(t)}.$$

$$\boxed{\arg\left(f'(\gamma_i(t))\gamma_i'(t)\right) = \arg\left(f'(\gamma_i(t))\right) + \arg\left(\gamma_i'(t)\right)}$$

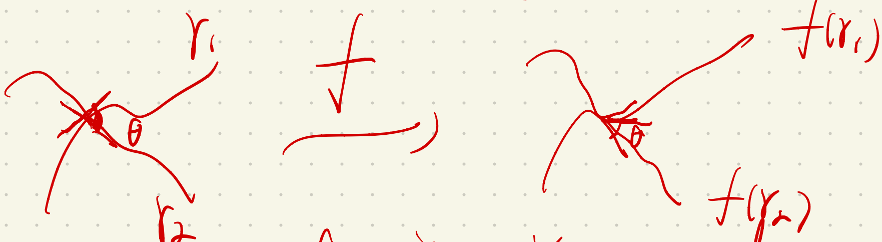
(We assume $f' \neq 0$ at $\gamma_i(t)$)

$$\arg\left(f'(\gamma_1(t_0))\right) - \arg\left(f'(\gamma_2(t_0))\right)$$

$$= \arg\left(f'(\gamma_1(t_0))\right) - \arg\left(f'(\gamma_2(t_0))\right) + \arg\left(\gamma_1'(t_0)\right) - \arg\left(\gamma_2'(t_0)\right).$$



$$= \arg\left(\gamma_1'(t_0)\right) - \arg\left(\gamma_2'(t_0)\right).$$



θ is preserved.